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The general correlation function in the Schwinger model on a torus

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Abstract
In the framework of the Euclidean path integral approach we derive the exact formula for the general N-point chiral densities correlator in the Schwinger model on a torus.

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1. Introduction

The Schwinger model [1] (SM) (two-dimensional QED with massless fermions) on a Euclidean torus $\mathcal{T}_2$ is exactly soluble [2–4], and in many calculations it would be useful to have an expression for the general N-point correlation function of chiral densities.

It is well known that in this model the 'photon' acquires a mass due to chiral anomaly and fermions disappear from the physical spectrum.

There are some features of the SM which are similar to those of QCD. Fermion condensate, mass generation, dynamical symmetry breaking and confinement are among them. In both models instantons are supposed to be responsible for some nontrivial vacuum expectation values [5–7].

Work on a torus is desirable for several reasons. Firstly, by defining the model on a finite volume we get rid of infrared problems. Compactification makes mathematical manipulations more rigorous, topological relations become more precise and transparent. One should remember that the fermion path integrals have no meaning unless defined using a discrete basis.

Secondly, in this case we have a model with nontrivial topology in which we can find explicitly fermionic zero modes and Green's functions in all topological sectors. The presence of topologically nontrivial configurations of the gauge field (instantons) and fermionic zero modes allows in path integral framework the reproduction of the structure of the SM found in the operator formalism [8].

Thirdly, a compactification on a torus allows us to find finite temperature and finite size effects and is appropriate to the systematic analysis of the lattice approximation. It is a torus which is most naturally approximated by the finite cubical lattice on which the numerical calculations are performed [9]. Interest in finite-temperature calculations in quantum field theories ($T \neq 0$ QFT) (or relativistic quantum statistics) is explained by the fact that there are some phenomena (early universe, ultrarelativistic heavy ion collisions, baryon number violating processes, etc) which should be described by $T \neq 0$ QFT. Another motivation is purely theoretical: consideration of the model at finite temperature (and finite size) can test its consistency and reasonableness and deepen one's understanding of its structure.

Finally, torus and circle are particularly appropriate for studying the relation between the Hamiltonian and path integral approach in the gauge theory with massless fermions [10, 11].

Thanks to its full solvability the SM may also be used to test various ideas related to nonperturbative structure of quantum field theory, in particular it is a good laboratory to investigate its topological aspects [12, 13].

Bardakci and Crescimanno were the first to use the path integral approach to explore the role of nontrivial topological configurations in the two-dimensional fermionic model relevant in the context of string compactification [14]. They were able to show that certain correlation functions which, being zero for the trivial topology, considerably change by the nontrivial topological effect. Following the ideas of Bardakci and Crescimanno, Manias, Naon and Trobo studied the behaviour of correlation functions of fermion bilinear operators in the SM with topologically nontrivial gauge configurations in the infinite spacetime [15]. Two- and four-point correlation functions in the SM on the torus have been calculated in [2, 16]. The authors of the paper [17] considered a six-point correlation function in this model which due to some technical difficulties they managed to calculate only at finite temperature but in the infinite space. They also made a conjecture about the explicit expression for the N-point correlator of chiral densities again at finite temperatures but in the infinite space.

The paper is organized as follows. In section 2, we briefly review the results previously obtained for the SM on the torus in the Euclidean (path integral) approach [2–4] and relevant for the present consideration. In addition to these results we give some new information which concerns the possible choice of the zero modes and fermionic Green's functions in the nontrivial topological sectors. Section 3 is devoted to the discussion of modular transformation. The invariance with respect to this transformation helps us in the following to determine some proportionality constants. In section 4, which is the central part of the present work, we obtain our main result. The last section is reserved for conclusions and the discussion of possible directions of the future investigation.

In the appendix we present some details of the thermodynamic and zero-temperature limits.

2. A brief review of path integral formulation of the SM on the Euclidean torus

The SM action on the Euclidean torus $\mathcal{T}_2$ $(0 \leqslant x_1 \leqslant L_1, 0 \leqslant x_2 \leqslant L_2)$ reads

$$S = \int_{\mathcal{T}_2} \mathrm{d}^2x \left(\tfrac{1}{2}F_{12}^2(x) + \bar{\psi}(x)\gamma_\mu(\partial_\mu - \mathrm{i}eA_\mu(x))\psi(x)\right) \tag{1}$$

where $F_{12}(x) = \partial_1 A_2(x) - \partial_2 A_1(x)$ is a field strength. We use the conventions and notation used in [3, 2]. The evaluation of quantum-mechanical expectation values (QMEV) in the path integral formulation

$$\langle \Omega[\bar{\psi}, \psi, A_\mu] \rangle = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}, A_\mu] \Omega[\bar{\psi}, \psi, A_\mu] \, \mathrm{e}^{-S[\bar{\psi}, \psi, A_\mu]} \tag{2}$$

where the Z factor (the partition function)

$$Z = \int \mathcal{D}[\psi, \bar{\psi}, A_\mu]\, \mathrm{e}^{-S[\bar{\psi},\psi,A_\mu]} \tag{3}$$

gets the following form for QMEV of fermion fields [2]:

$$\begin{aligned}
&\left\langle \psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_1}(y_1)\cdots\psi_{\alpha_N}(x_N)\bar{\psi}_{\beta_N}(y_N)\right\rangle \\
&\quad= Z^{-1}\sum_{k=0,\pm1,\dots,\pm N} L_1^{|k|}\int_{\mathcal{A}_k}\mathcal{D}A\,\mathrm{e}^{-S[A]}\det{}'[L_1\gamma_\mu(\partial_\mu - \mathrm{i}eA_\mu)] \\
&\qquad\times\sum_{P_i}(-1)^{p_i}\sum_{P_j}(-1)^{p_j}\hat{\chi}^{(1)}_{\alpha_{i_1}}\left(x_{i_1}\right)\cdots\hat{\chi}^{(|k|)}_{\alpha_{i_{|k|}}}\left(x_{i_{|k|}}\right)\bar{\hat{\chi}}^{(1)}_{\beta_{j_1}}(y_{j_1})\cdots\bar{\hat{\chi}}^{(|k|)}_{\beta_{j_{|k|}}}\left(y_{j_{|k|}}\right) \\
&\qquad\times S^{(k)}_{\alpha_{i_{|k|+1}}\beta_{j_{|k|+1}}}\left(x_{i_{|k|+1}}, y_{j_{|k|+1}}; A\right)\cdots S^{(k)}_{\alpha_{i_N}\beta_{j_N}}\left(x_{i_N}, y_{j_N}; A\right).
\end{aligned} \tag{4}$$

Here we have already performed the fermion integration $\int\mathcal{D}[\psi,\bar{\psi}]$ over the fermionic Grassmann variables. The sum is taken over all possible permutations $P_i = (i_1, i_2, \dots, i_N)$ and $P_j = (j_1, j_2, \dots, j_N)$ of $(1, 2, \dots, N)$, $(-1)^{p_i}$ $((-1)^{p_j})$ is a parity of the permutation P_i (P_j). $\hat{\chi}^{(n)}(x)$, $\bar{\hat{\chi}}^{(n)}(y)$, $n = 1, \dots, |k|$ is an orthonormal set of the zero-mode wavefunctions and $S^{(k)}(x, y; A)$ is a Green's function of the dimensionless $D[A] = L_1\gamma_\mu(\partial_\mu - \mathrm{i}eA_\mu)$ operator with a gauge field A_μ from the topological sector $\mathcal{A}_k$, $\det'[D(A)]$ is a product of its nonzero eigenvalues. In the following subsections these objects will be described in detail.

Let us discuss the topological origin and the physical meaning of different parts of this expression for the SM on the 2d torus.

2.1. Topology of $U(1)$-gauge fields $A_\mu(x)$ on the torus

The topology of $U(1)$-gauge fields on $\mathcal{T}_2$:

$$A_\mu(x) = C^{(k)}_\mu(x) + t_\mu + \epsilon_{\mu\nu}\partial_\nu b(x) + \partial_\mu a(x) \tag{5}$$

is given by the decomposition of A_μ into Chern classes together with the Hodge decomposition [18]. $C^{(k)}_\mu(x) = -\frac{\pi k}{eL_1L_2}\epsilon_{\mu\nu}x_\nu$, a gauge potential in the Lorentz gauge which leads to a constant field strength $F_{\mu\nu}(x) = \frac{2\pi k}{eL_1L_2}\epsilon_{\mu\nu}$ of the stationary gauge action. It belongs to the Chern class with a topological charge (topological quantum number) $\frac{e}{2\pi}\int_{\mathcal{T}_2}F_{12}\,\mathrm{d}^2x = k$, and plays the role of an *instanton* in our model. It defines a connection of a principal nontrivial $U(1)$-bundle over $\mathcal{T}_2$ with transition functions $\Lambda_\nu(x)$:

$$A_\mu(x + \hat{L}_\nu) = A_\mu(x) - \frac{\mathrm{i}}{e}\Lambda_\nu^{-1}(x)\partial_\mu\Lambda_\nu(x).$$

In our gauge the transition functions are gauge transformations:

$$\Lambda_1(x) = \mathrm{e}^{\pi\mathrm{i}k\frac{x_2}{L_2}} \qquad \Lambda_2(x) = \mathrm{e}^{-\pi\mathrm{i}k\frac{x_1}{L_1}} \tag{6}$$

and describe the continuation of $C^{(k)}$ in the $U(1)$-bundles along a cycle in $\mathcal{T}_2$. t_μ is a harmonic potential: $\Box t_\mu = 0$, called a toron field. It is a zero mode of the gauge field and is restricted to $0 \leqslant t_\mu < T_\mu$, where $T_\mu \equiv \frac{2\pi}{eL_\mu}$. $\epsilon_{\mu\nu}\partial_\nu b(x)$ describes gauge-independent 'deformations' of $C^{(k)}_\mu(x)$, $a(x)$ is a pure local gauge: $\partial_\mu a(x) = -\frac{\mathrm{i}}{e}\,\mathrm{e}^{-\mathrm{i}ea(x)}\partial_\mu\,\mathrm{e}^{\mathrm{i}ea(x)}$. *Large gauge transformations* on $\mathcal{T}_2$: $\Lambda(x) = \exp 2\pi\mathrm{i}\left(m_1\frac{x_1}{L_1} + m_2\frac{x_2}{L_2}\right)$ transform the toron field according to $t_\mu \to t_\mu + T_\mu m_\mu$. The Hodge decomposition leads to a product decomposition of the functional measure appearing in the path integral formulation

$$\int\mathcal{D}A = \sum_k\int\mathcal{D}A^{(k)} = \sum_k\int_0^{T_\mu}\mathrm{d}t_\mu\int\mathcal{D}a\int\mathcal{D}b. \tag{7}$$

2.2. Fermionic zero modes $\chi(x)$

The Atiyah–Singer index theorem for Dirac equation [18] states that the number of solutions with spin parallel minus the number of solutions with spin anti-parallel is equal to $|k|$. The fermionic zero modes are solutions of the Dirac equation satisfying the periodic boundary conditions described by the transition functions $\Lambda_\nu(x)$ of the $U(1)$-bundle:

$$\gamma_\mu(\partial_\mu - \mathrm{i}eA_\mu)\hat{\chi}(x) = 0 \qquad \text{with} \quad \hat{\chi}(x+\hat{L}_\nu) = \Lambda_\nu(x)\hat{\chi}(x). \tag{8}$$

In future we will also consider the operator

$$D_0 = D|_{a=b=0} = \gamma_\mu\left(\partial_\mu - \mathrm{i}e\left(t_\mu + C_\mu^{(k)}\right)\right). \tag{9}$$

Then $\hat{\chi}^{(j)}(x) = \mathrm{e}^{\mathrm{i}ea(x)+e\gamma_5 b(x)}\chi^{(j)}(x)$, where $\chi^{(j)}(x)$ is a zero mode of the D_0 operator, which can be explicitly expressed by Jacobi's θ-functions [19, 20]. The most general expression for the zero modes of the D_0 operator with positive chirality ($k > 0$) in the Lorentz gauge takes the form [21] ($j = 1, \ldots, k$)

$$\chi^{(j)}(x) = \begin{pmatrix} \chi_1^{(j)}(x) \\ 0 \end{pmatrix} \tag{10}$$

with

$$\chi_1^{(j)}(x) = \left(\frac{2k}{|\tau|}\right)^{1/4} \frac{1}{L_1}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{|\tau|}\zeta\bar{t} + \frac{\mathrm{i}\pi k}{|\tau|}z\zeta - \frac{\mathrm{i}\pi}{k}\bar{t}\tilde{t}_1}\, T_k^{(j)}(z') \tag{11}$$

where functions $T_k^{(j)}(z)$ obey the periodicity conditions

$$T_k^{(j)}(z+1) = T_k^{(j)}(z) \qquad T_k^{(j)}(z+\tau) = \mathrm{e}^{-\mathrm{i}\pi k(2z+\tau)}T_k^{(j)}(z) \tag{12}$$

and are chosen in such a way that the zero modes (11) are orthonormalized ($z' \equiv z + \bar{t}/k$, where $z \equiv \frac{x_1+\mathrm{i}x_2}{L_1}$, $\zeta \equiv \operatorname{Im} z$, $\tau \equiv \mathrm{i}\frac{L_2}{L_1}$ and $t \equiv \tilde{t}_2 + \mathrm{i}|\tau|\tilde{t}_1$, $\tilde{t}_\mu \equiv \frac{eL_\mu}{2\pi}t_\mu$, a bar means complex conjugation). The constant factor on the rhs of equation (11) is chosen for convenience. We have two explicit solutions. One was presented in [2]

$$T_k^{(j)}(z') = \mathrm{e}^{-\frac{\pi(j-1)^2}{k}|\tau| + 2\pi\mathrm{i}(j-1)z'}\theta_3(kz' + (j-1)\tau|k\tau). \tag{13}$$

Another has the form

$$T_k^{(j)}(z') = \frac{1}{\sqrt{k}}\theta_3\left(z' - \frac{(j-1)}{k}\,\Big|\,\frac{\tau}{k}\right). \tag{14}$$

Note that this is actually the solution found by Sachs and Wipf [4]. In order to see this one should apply to it the modular transformation considered in section 3.

Of course, each function of the set (14) is a linear combination of the functions of the set (13), since there is a relation

$$\theta_3(z|\tau/k) = \sum_{l=0}^{k-1} \mathrm{e}^{\pi\mathrm{i}l^2\frac{\tau}{k} + 2\pi\mathrm{i}lz}\theta_3(kz + l\tau|k\tau). \tag{15}$$

For the zero modes of negative chirality ($k < 0$) we have

$$\phi^{(j)}(x) = \begin{pmatrix} 0 \\ \phi_2^{(j)}(x) \end{pmatrix} \qquad j = 1, \ldots, |k| \tag{16}$$

with

$$\phi_2^{(j)}(x) = \left(\frac{2|k|}{|\tau|}\right)^{1/4} \frac{1}{L_1}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{|\tau|}\zeta t - \frac{\mathrm{i}\pi|k|}{|\tau|}\bar{z}\zeta + \frac{\mathrm{i}\pi}{|k|}t\tilde{t}_1}\, T_{|k|}^{(j)}(\bar{z}'') \tag{17}$$

where $\bar{z}'' \equiv \bar{z} - \frac{t}{|k|}$.

2.3. *Regularized effective action* $\Gamma^{(k)}_{\text{reg}}[A]$

We have calculated the regularized effective action
$\Gamma^{(k)}_{\text{reg}}[A] = 2\ln\det'(L_1\gamma_\mu(\partial_\mu - \mathrm{i}eA_\mu)) + \Gamma_{\text{reg}}(\{M_j\})$ for different topological sectors. The result is [2, 3]

$$\Gamma^{(k)}_{\text{reg}}[A] = \frac{e^2}{\pi}\int_{\mathcal{T}_2} \mathrm{d}^2x\, b(x)\,\Box\, b(x) + 2\delta_{0,k}\ln\left|\,\mathrm{e}^{-2\pi|\tau|\bar{t}_1^2}\theta_1(t|\tau)\eta^{-1}(\tau)\right|^2$$
$$- (1-\delta_{0,k})\left\{2\ln\det\mathcal{N}^{(k)}_A - |k|(\ln(2|k|/|\tau| - 2\pi\mathrm{i}))\right\} + \Gamma_{\text{reg}}(\{M_j\}). \qquad (18)$$

As discussed below the first term is a 'mass term', see equation (25). The second term defines the 'induced toron action' $\Gamma^{(0)}[\mathrm{t}]$ ($\eta(\tau)$ is Dedekind's function). It is induced by the fermions via the spectral flow of the Dirac operator [11]. In calculating the effective action for gauge fields from the topological nontrivial sectors $k \neq 0$, one has to separate the zero modes. The third line contains the determinant of the matrix of the scalar products of the (non-orthonormal) zero modes $\mathcal{N}^{(k)}_A$, and a weight factor of the nontrivial sectors [3, 2]. The regularization term $\Gamma_{\text{reg}}\{(M_j)\}$ drops off by the normalization of the path integral formula. The term $|k|(\ln(2|k|/|\tau| - 2\pi\mathrm{i})$ compensates the length scale dependence of the zero mode normalization. It determines the relative weights of the contributions from different topological sectors. Observe that in the general formula (4) it is assumed that zero modes $\hat{\chi}^{(j)}(x)$ are orthonormalized. If not, the matrix $\mathcal{N}^{(k)}_A$ will enter this formula (see equation (51)).

2.4. *The fermion propagator* $S^{(k)}(x, y; A)$

It follows from the well-known solution of the 2d Dirac equation with external gauge potential that the fermion propagator can be written as

$$S^{(k)}(x, y; A) = \mathrm{e}^{\mathrm{i}e\alpha(x)}S^{(k)}_t(x, y)\,\mathrm{e}^{-\mathrm{i}e\alpha^\dagger(y)} \qquad (19)$$

with $\alpha(x) = a(x) - \mathrm{i}\gamma_5 b(x)$, where $S^{(k)}_t(x, y)$ is a propagator of fermions in the background gauge field $A_\mu(x), a = b = 0$ from the sector with the topological charge k. There is an explicit expression for $S^{(0)}_t(x, y)$ in terms of θ-functions[1] [2, 3, 16]:

$$S^{(0)}_t(x-y) = \begin{pmatrix} 0 & \frac{\eta^3}{L_1}\frac{\theta_1(z-w+\bar{t})}{\theta_1(\bar{t})\theta_1(z-w)}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{|\tau|}(\zeta-\xi)\bar{t}} \\ -\frac{\eta^3}{L_1}\frac{\theta_1(\bar{z}-\bar{w}-t)}{\theta_1(t)\theta_1(\bar{z}-\bar{w})}\,\mathrm{e}^{\frac{-2\pi\mathrm{i}}{|\tau|}(\zeta-\xi)t} & 0 \end{pmatrix} \qquad (20)$$

where $w \equiv \frac{y_1+\mathrm{i}y_2}{L_1}$ and $\xi \equiv \operatorname{Im} w$. Note that $S^{(0)}_t(x)$ becomes singular for $t = 0$. This singularity is caused by the constant solution of the Dirac equation with $t = 0$. It represents a zero mode in the trivial sector. In the path integral it is compensated by a zero of the Boltzmann factor of the induced toron action: $\sim\exp(\Gamma^{(0)}[t]/2)$. In the sector with $k > 0$ the fermionic Green function takes a form [2, 17]

$$S^{(k)}_t(x, y) = S^{(0)}_t(x, y)\frac{q^{(k)}(z)}{q^{(k)}(w)}\,\mathrm{e}^{\frac{\mathrm{i}\pi k}{|\tau|}(z\zeta - w\xi)} \qquad (21)$$

where $q^{(k)}(z)$ is a function which obeys the same periodicity conditions as the functions $T^{(j)}_k(z)$ (see equation (12))

$$q^{(k)}(z+1) = q^{(k)}(z) \qquad q^{(k)}(z+\tau) = \mathrm{e}^{-\mathrm{i}\pi k(2z+\tau)}q^{(k)}(z) \qquad (22)$$

[1] In what follows we will use shorthand notation for θ-functions $\theta_\alpha(z) \equiv \theta_\alpha(z|\tau)$ if the second argument of a theta function is τ.

and has no poles in z. Some examples are as follows:

$$q^{(k)}(z) = \theta_3(z|\tau/k) \tag{23}$$

$$q_l^{(k)}(z) = \mathrm{e}^{2\pi \mathrm{i} l z}\theta_3(kz + l\tau|k\tau) \tag{24}$$

where $l = 0, 1, \ldots, k-1$. Note that any linear combination of functions (24) also obeys the periodicity conditions (22). When $k \neq 0$ the choice of the fermionic Green function is not unique. All possible choices differ by the linear combination of the zero modes.

2.5. Scalar propagators on the torus

The $b(x)$-dependent part of the action consists of the gauge field action $S_g[A]$ and the mass term of $\Gamma^{(k)}_{\mathrm{reg}}[A]$ giving together $S[b] = 1/2\int_{\mathcal{T}_2} \mathrm{d}x\, b(x)\, \Box\, (\Box - m^2) b(x)$ with $m^2 \equiv e^2/\pi$. The corresponding propagator satisfies the equation

$$\Box(\Box - m^2) G(x-y) = \delta^{(2)}(x-y) - \frac{1}{L_1 L_2} \tag{25}$$

where $\delta^{(2)}(x-y)$ is Dirac's δ-function on the torus. It can be written as the difference of a massless and massive propagator on the torus orthogonal to the constant functions: $G(x) = 1/m^2\{G_0(x) - G_m(x)\}$. There is a closed expression in the massless case

$$G_0(x) = -\frac{1}{2\pi}\ln\left(\eta^{-1}(\tau)\,\mathrm{e}^{-\frac{\pi\zeta^2}{|\tau|}}|\theta_1(z)|\right). \tag{26}$$

In the massive case we use the infinite sum for $\overline{G}_m(x) = G_m(x) + 1/m^2 L_1 L_2$:

$$\overline{G}_m(x) = \frac{1}{2L_1}\sum_n \frac{\cosh[E(n)(L_2/2 - |x_2|)]\,\mathrm{e}^{2\pi \mathrm{i} n \frac{x_1}{L_1}}}{E(n)\sinh[L_2 E(n)/2]} \tag{27}$$

where

$$E(n) = \sqrt{4\pi^2 n^2 L_1^{-2} + m^2}.$$

2.6. Chiral condensate and two-point correlators of chiral densities

If one considers QMEV only of gauge-invariant quantities the pure gauge field $a(x)$ may be integrated over with no consequence and we will not consider it in future. Then in the topological sector with the topological charge k we may write

$$\int_{\mathcal{A}_k} \mathcal{D}A\,\mathrm{e}^{-S[A]}\ldots = \mathrm{e}^{-\frac{2\pi k^2}{m^2 L_1 L_2}}\int_0^{T_1}\mathrm{d}t_1\int_0^{T_2}\mathrm{d}t_2\int\mathcal{D}b\,\mathrm{e}^{-\frac{1}{2}\int b(x)\Box^2 b(x)\,\mathrm{d}^2x}\ldots. \tag{28}$$

The partition function (3) is a product of three factors:

$$Z = \int_{\mathcal{A}_0}\mathcal{D}A\,\mathrm{e}^{-S[A]+\frac{1}{2}\Gamma^{(0)}_{\mathrm{reg}}[A]} = Z_0 Z_t Z_M \tag{29}$$

where

$$Z_0 = \int\mathcal{D}b\,\mathrm{e}^{-\frac{1}{2}\int\mathrm{d}^2x\, b(x)\Box\left(\Box - m^2\right)b(x)} \tag{30}$$

$$Z_t = \int_0^{T_1}\mathrm{d}t_1\int_0^{T_2}\mathrm{d}t_2\,\mathrm{e}^{\frac{1}{2}\Gamma^{(0)}[t]} = \frac{(2\pi)^2}{e^2\sqrt{2|\tau|}L_1 L_2\eta^2(\tau)} \tag{31}$$

$$Z_M = \mathrm{e}^{\frac{1}{2}\Gamma_{\mathrm{reg}}(\{M_j\})}. \tag{32}$$

The QMEV of fermion fields exhibit the mechanism of chiral symmetry breaking by an anomaly. The configurations with $k = \pm 1$ are responsible for the formation of the chiral condensate [3, 4, 2], which has the form

$$\langle \bar{\psi}(x) P_{\pm} \psi(x) \rangle = -\frac{\eta^2(\tau)}{L_1} \, \mathrm{e}^{2e^2 G(0) - \frac{2\pi^2}{e^2 L_1 L_2}} \tag{33}$$

where $P_{\pm} \equiv \frac{1}{2}(1 \pm \gamma_5)$, and two-point correlators of chiral densities

$$\langle \bar{\psi}(x) P_{+} \psi(x) \bar{\psi}(y) P_{-} \psi(y) \rangle = (\langle \bar{\psi}(x) P_{+} \psi(x) \rangle)^2 \, \mathrm{e}^{4\pi \overline{G}_m(x-y)} \tag{34}$$

$$\langle \bar{\psi}(x) P_{+} \psi(x) \bar{\psi}(y) P_{+} \psi(y) \rangle = (\langle \bar{\psi}(x) P_{+} \psi(x) \rangle)^2 \, \mathrm{e}^{-4\pi \overline{G}_m(x-y)} \tag{35}$$

get contributions from the topological sectors with $k = 0$ and $k = 2$, respectively [2, 16]. The expressions for these correlators of the SM in the infinite spacetime were first obtained by Casher *et al* [22] who used the bosonization techniques in the operator formalism. The nonvanishing $\langle \bar{\psi}(x)\psi(x) \rangle^2$ in this model was obtained for the first time by Lowenstein and Swieca [8]. Sachs and Wipf [4] calculated the gauge-invariant chiral two-point function

$$S_{\pm}(x, y) = \left\langle \bar{\psi}(x) \, \mathrm{e}^{\mathrm{i}e \int_y^x A_\alpha(\xi) \mathrm{d}\xi_\alpha} P_{\pm} \psi(y) \right\rangle$$

which may be related to a bound state between a static external charge and the dynamical fermion. In this case only the sector with topological charge $|k| = 1$ contributes. A more general gauge-invariant fermion two-point function

$$S_{\alpha,\beta}(x, 0) = -\left\langle \psi_\alpha(0) \, \mathrm{e}^{\mathrm{i}e \int_0^x A_\mu(\xi) \mathrm{d}\xi_\mu} \bar{\psi}_\beta(x) \right\rangle$$

was calculated in [7]. The gauge-invariant correlator offers a suitable framework for probing both chiral symmetry breaking and confinement (screening) at zero and finite temperatures through its short and large distance limits. In [13], the Euclidean path integral representation for tunnelling Green's functions

$$\langle n | \psi(x_1) \cdots \psi(x_n) \bar{\psi}(y_1) \cdots \bar{\psi}(y_n) | 0 \rangle$$

has been derived using clustering arguments in the infinite spacetime (here $|n\rangle$ is a vacuum state with the winding number n).

As was mentioned above in the SM the chiral symmetry breaking occurs due to the anomaly. This phenomenon may be relevant to the U(1) problem in QCD, although by its nature the breakdown of the dynamical chiral symmetry in QCD which allows, e.g. to consider the pion as a Goldstone boson is different.

Furthermore, the zero modes which play the crucial role in the formation of the chiral condensate in the SM are irrelevant in the case of one flavour QCD with a small quark mass where a quark condensate could be estimated via the Banks–Casher formula [23].

3. Modular transformation

Under exchange

$$(x_i)_1 \leftrightarrow (x_i)_2 \qquad L_1 \leftrightarrow L_2 \qquad t_1 \leftrightarrow t_2 \qquad \gamma_1 \leftrightarrow \gamma_2 \tag{36}$$

we have a modular transformation

$$\tau \to -\frac{1}{\tau} \qquad z \to -\frac{\bar{z}}{\tau} \qquad \bar{z} \to \frac{z}{\tau} \qquad t \to -\frac{\bar{t}}{\tau} \qquad \zeta \to \frac{\bar{z} + z}{2|\tau|} \qquad \gamma_5 \to -\gamma_5. \tag{37}$$

Under this transformation we have transitions

$$
\begin{aligned}
&\eta(\tau) \to \eta\left(-\frac{1}{\tau}\right) = \sqrt{|\tau|}\eta(\tau) \\
&\theta_1(z|\tau) \to \theta_1\left(-\frac{\bar{z}}{\tau}\bigg| -\frac{1}{\tau}\right) = \mathrm{i}\sqrt{|\tau|}\,\mathrm{e}^{\frac{\mathrm{i}\pi\bar{z}^2}{\tau}}\theta_1(\bar{z}|\tau) \\
&\theta_1(z+\bar{t}|\tau) \to \theta_1\left(-\frac{\bar{z}}{\tau}+\frac{t}{\tau}\bigg| -\frac{1}{\tau}\right) = \sqrt{|\tau|}\,\mathrm{e}^{\frac{\pi\bar{z}^2}{|\tau|}-\frac{2\pi\bar{z}t}{|\tau|}+\frac{\pi t^2}{|\tau|}}\theta_1(\bar{z}-t|\tau).
\end{aligned} \tag{38}
$$

One can easily check that under modular transformation: $G_0(x)$ and $S_t^{(0)}(x)$ are invariant, there is an exchange of the zero modes of opposite chirality: $\chi_1^{(j)}(x) \leftrightarrow \phi_2^{(j)}(x)$ and

$$
\mathrm{e}^{\frac{\mathrm{i}\pi k}{|\tau|}z\zeta}\theta_3(z|\tau/k) \to \sqrt{|\tau|k}\,\mathrm{e}^{-\frac{\mathrm{i}\pi k}{|\tau|}\bar{z}\zeta}\theta_3(k\bar{z}|k\tau) \tag{39}
$$

$$
\mathrm{e}^{\frac{\mathrm{i}\pi k}{|\tau|}z\zeta}\theta_3(kz|k\tau) \to \sqrt{\frac{|\tau|}{k}}\,\mathrm{e}^{\frac{\mathrm{i}\pi k}{|\tau|}\bar{z}\zeta}\theta_3(\bar{z}|\tau/k). \tag{40}
$$

So the modular transformation acts on the fermionic Green function in the nontrivial topological sector (21) effectively (up to a linear combination of zero modes) as a complex conjugation.

4. General formula

The general formula which we want to prove is

$$
\left\langle \prod_{i=1}^{N} \bar{\psi}(x_i)P_{e_i}\psi(x_i)\right\rangle = (\langle\bar{\psi}(x)P_+\psi(x)\rangle)^N\,\mathrm{e}^{-4\pi\sum_{i<j}e_ie_j\overline{G}_m(x_i-x_j)} \tag{41}
$$

where $e_i = \pm(\pm 1)$.

Without loss of generality we may consider two cases ($N = r+s$, $r-s = k$, $s \leqslant r$, $r(s)$ is a number of factors in the lhs of equation (41) with $e = +(e = -)$.

(1) $r = s$. Only a trivial sector $k = 0$ contributes. From the most general formula (4) it follows that

$$
\left\langle \prod_{i=1}^{r} \bar{\psi}(x_i)P_+\psi(x_i)\bar{\psi}(y_i)P_-\psi(y_i)\right\rangle = Z^{-1}\int_{\mathcal{A}_0}\mathcal{D}A\,\mathrm{e}^{-S[A]+\frac{1}{2}\Gamma^{(0)}_{\mathrm{reg}}[A]}\left|\det\left\|S^{(0)}_{12}(x_i,y_i;A)\right\|\right|^2. \tag{42}
$$

(We omit the matrix indices 12 of the 2×2 fermion propagator matrix in the following for shorthand. Since we consider only the case with $k \geqslant 0$ only this matrix element will be appearing in our calculations.) With the help of equation (19) we may write

$$
|\det\|S(x_i,y_j;A)\||^2 = \mathrm{e}^{2e\sum_{i=1}^{r}[b(x_i)-b(y_i)]}\left|\det\left\|S_t^{(0)}(x_i,y_j)\right\|\right|^2 \tag{43}
$$

and do the path integration over the b field with the result

$$
\begin{aligned}
\left\langle \prod_{i=1}^{r} \bar{\psi}(x_i)P_+\psi(x_i)\bar{\psi}(y_i)P_-\psi(y_i)\right\rangle &= Z_t^{-1}\,\mathrm{e}^{N\left(2e^2G(0)-\frac{2\pi^2}{e^2L_1L_2}\right)} \\
&\times \mathrm{e}^{-4\pi\sum_{i<i'}^{r}\overline{G}_m(x_i-x_{i'})-4\pi\sum_{j<j'}^{r}\overline{G}_m(y_j-y_{j'})+4\pi\sum_{i=1}^{r}\sum_{j=1}^{r}\overline{G}_m(x_i-y_j)} \\
&\times \mathrm{e}^{4\pi\sum_{i<i'}^{r}G_0(x_i-x_{i'})+4\pi\sum_{j<j'}^{r}G_0(y_j-y_{j'})-4\pi\sum_{i=1}^{r}\sum_{j=1}^{r}G_0(x_i-y_j)} \\
&\times \int_0^{T_1}\mathrm{d}t_1\int_0^{T_2}\mathrm{d}t_2\left|\det\left\|S_t^{(0)}(x_i,y_j)\right\|\right|^2\left|\theta_1(\bar{t})\right|^2\mathrm{e}^{-2\pi|\tau|\bar{t}_1^2}\eta^{-2}.
\end{aligned} \tag{44}
$$

Then the only integration which is left is the integration with respect to the toron field. From equation (20) it follows ($i, j = 1, \ldots, r$) that

$$\det \left\| S_t^{(0)}(x_i, y_j) \right\| = \left(\frac{\eta^3}{L_1} \right)^r \mathrm{e}^{\frac{2\pi \mathrm{i}}{|\tau|} \sum_{i=1}^r (\zeta_i - \xi_i) \bar{t}} \det \left\| \frac{\theta_1(z_i - w_j + \bar{t})}{\theta_1(z_i - w_j)\theta_1(\bar{t})} \right\|. \tag{45}$$

It can be proved that

$$\det \left\| \frac{\theta_1(z_i - w_j + \bar{t})}{\theta_1(z_i - w_j)\theta_1(\bar{t})} \right\| = (-1)^{\frac{r(r-1)}{2}} \frac{\prod_{i<j}^r \theta_1(z_i - z_j)\theta_1(w_i - w_j)}{\theta_1(\bar{t}) \prod_{i,j}^r \theta_1(z_i - w_j)} \theta_1 \left(\sum_{i=1}^r (z_i - w_i) + \bar{t} \right). \tag{46}$$

This formula is a generalization to the torus of the Cauchy determinant formula [24]

$$\det \left\| \frac{1}{z_i - w_j} \right\| = (-1)^{\frac{r(r-1)}{2}} \frac{\prod_{i<j}^r (z_i - z_j)(w_i - w_j)}{\prod_{i,j}^r (z_i - w_j)}. \tag{47}$$

The proof is based on the examination of the zero and pole structure of the lhs of equation (46) using short-distance behaviour given in the lhs of equation (47). Then one may check that functions on both sides obey the same periodicity conditions when $z_i \to z_i + 1$, $w_j \to w_j + 1$ and $z_i \to z_i + \tau$, $w_j \to w_j + \tau$ (standard elliptic function arguments).

Now the integration with respect to the toron field can be done with the help of the formula

$$\int_0^1 \mathrm{d}\tilde{t}_1 \int_0^1 \mathrm{d}\tilde{t}_2 \, \mathrm{e}^{4\pi \zeta \tilde{t}_1} \theta_a(z + \bar{t}) \theta_a(\bar{z} + t) \, \mathrm{e}^{-2\pi|\tau|\tilde{t}_1^2} = \frac{\mathrm{e}^{\frac{2\pi\zeta^2}{|\tau|}}}{\sqrt{2|\tau|}} \qquad a = 1, 3 \tag{48}$$

and we obtain using equation (31)

$$\begin{aligned} & Z_t^{-1} \int_0^{T_1} \mathrm{d}t_1 \int_0^{T_2} \mathrm{d}t_2 \left| \det \left\| S_t^{(0)}(x_i, y_j) \right\| \right|^2 |\theta_1(\bar{t})|^2 \, \mathrm{e}^{-2\pi|\tau|\tilde{t}_1^2} \eta^{-2} \\ & \quad = \left(\frac{\eta^3}{L_1} \right)^{2r} \frac{1}{\sqrt{2|\tau|}} \, \mathrm{e}^{-\frac{2\pi}{|\tau|} \{ \sum_{i<j}^r [(\zeta_i - \zeta_j)^2 + (\xi_i - \xi_j)^2] - \sum_{i=1}^r \sum_{j=1}^r (\zeta_i - \xi_j)^2 \}} \\ & \qquad \times \frac{\prod_{i<i'}^r |\theta_1(z_i - z_{i'})|^2 \prod_{j<j'}^r |\theta_1(w_j - w_{j'})|^2}{\prod_{i=1}^r \prod_{j=1}^r |\theta_1(z_i - w_j)|^2}. \end{aligned} \tag{49}$$

If we insert this result into equation (44) and take into account equation (26) together with equation (33) we will see that this is equation (41) for the case when $r = s$.

(2) $r - s = k > 0$. Only a sector with the topological charge k contributes and equation (41) takes the form

$$\begin{aligned} & \left\langle \prod_{i=1}^r \bar{\psi}(x_i) P_+ \psi(x_i) \prod_{j=1}^s \bar{\psi}(y_j) P_- \psi(y_j) \right\rangle = (\langle \bar{\psi}(x) P_+ \psi(x) \rangle)^N \\ & \qquad \times \mathrm{e}^{-4\pi \sum_{i<i'}^r \overline{G}_m(x_i - x_{i'}) - 4\pi \sum_{j<j'}^s \overline{G}_m(y_j - y_{j'}) + 4\pi \sum_{i=1}^r \sum_{j=1}^s \overline{G}_m(x_i - y_j)}. \end{aligned} \tag{50}$$

From the general formula (4) for $s \geqslant 1$ it follows that

$$\begin{aligned} & \left\langle \prod_{i=1}^r \bar{\psi}(x_i) P_+ \psi(x_i) \prod_{j=1}^s \bar{\psi}(y_j) P_- \psi(y_j) \right\rangle \\ & \qquad = Z^{-1} L_1^k \int_{\mathcal{A}_k} \mathcal{D}A \, \mathrm{e}^{-S[A] + \frac{1}{2}\Gamma_{\mathrm{reg}}^{(k)}[A]} |\det \| (\hat{\chi}, S^{(k)}) \| |^2 \left(\det \mathcal{N}_A^{(k)} \right)^{-1} \end{aligned} \tag{51}$$

where the $r \times r$ matrix $\|(\hat{\chi}, S^{(k)})\|$ reads

$$\|(\hat{\chi}, S^{(k)})\| = \begin{pmatrix} \hat{\chi}_1^{(1)}(x_1) & \dots & \hat{\chi}_1^{(k)}(x_1) & S^{(k)}(x_1, y_1; A) & \dots & S^{(k)}(x_1, y_s; A) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hat{\chi}_1^{(1)}(x_r) & \dots & \hat{\chi}_1^{(k)}(x_r) & S^{(k)}(x_r, y_1; A) & \dots & S^{(k)}(x_r, y_s; A) \end{pmatrix}. \tag{52}$$

Now we may use the relation between the zero modes $\hat{\chi}(x)$ and $\chi(x)$, equation (19), and do the path integration over the $b(x)$ field with the result

$$\begin{aligned} \left\langle \prod_{i=1}^{r} \bar{\psi}(x_i) P_+ \psi(x_i) \prod_{j=1}^{s} \bar{\psi}(y_j) P_- \psi(y_j) \right\rangle &= Z_t^{-1}\, \mathrm{e}^{N\left(2e^2 G(0) - \frac{2\pi^2}{e^2 L_1 L_2}\right)} \\ &\times \mathrm{e}^{-4\pi \sum_{i<i'}^{r} \overline{G}_m(x_i - x_{i'}) - 4\pi \sum_{j<j'}^{s} \overline{G}_m(y_j - y_{j'}) + 4\pi \sum_{i=1}^{r}\sum_{j=1}^{s} \overline{G}_m(x_i - y_j)} \\ &\times \mathrm{e}^{4\pi \sum_{i<i'}^{r} G_0(x_i - x_{i'}) + 4\pi \sum_{j<j'}^{s} G_0(y_j - y_{j'}) - 4\pi \sum_{i=1}^{r}\sum_{j=1}^{s} G_0(x_i - y_j)} \\ &\times \int_0^{T_1} \mathrm{d}t_1 \int_0^{T_2} \mathrm{d}t_2 \left| \det \left\| \left(\chi, S_t^{(k)}\right) \right\| \right|^2. \end{aligned} \tag{53}$$

From equation (21) it follows that

$$\begin{aligned} \left| \det \left\| \left(\chi, S_t^{(k)}\right) \right\| \right|^2 &= \left(\frac{2k}{|\tau|}\right)^{k/2} \frac{\eta^{6s}}{L_1^{2k+2s}}\, \mathrm{e}^{4\pi \bar{t}_1 \left(\sum_{i=1}^{r} \zeta_i - \sum_{j=1}^{s} \xi_j\right)} \\ &\times \mathrm{e}^{-\frac{2\pi k}{|\tau|}\left(\sum_{i=1}^{r} \zeta_i^2 - \sum_{j=1}^{s} \xi_j^2\right) - 2\pi|\tau|\bar{t}_1^2} |\det \|M_{i,j}\||^2 \end{aligned} \tag{54}$$

where the $r \times r$ matrix $\|M_{ij}\|$ is defined such that for $1 \leqslant j \leqslant k$

$$M_{ij} = T_k^{(j)}(z_i + \bar{t}/k) \tag{55}$$

and for $k + 1 \leqslant j \leqslant r$

$$M_{ij} = \frac{\theta_1(z_i - w_{j-k} + \bar{t})\, q^{(k)}(z_i)}{\theta_1(z_i - w_{j-k})\theta_1(\bar{t})\, q^{(k)}(w_{j-k})}. \tag{56}$$

Now for the determinant of the matrix M the following expression can be proved:

$$\det \|M_{i,j}\| = C_k \frac{\prod_{i<i'}^{r} \theta_1(z_i - z_{i'}) \prod_{j<j'}^{s} \theta_1(w_j - w_{j'})}{\prod_{i=1}^{r}\prod_{j=1}^{s} \theta_1(z_i - w_j)} \theta_a \left(\sum_{i=1}^{r} z_i - \sum_{j=1}^{s} w_j + \bar{t} \right) \tag{57}$$

where $a = 1$ (3) if k is even (odd). (If $s = 1$ one should take 1 instead of the product $\prod_{j<j'}$ in the numerator.) For the cases $r = 2, s = 1$ (considered in [17]) and $r = 3, s = 1$ we calculated the determinant on the lhs explicitly using equations (11), (20) and (21) and checked the formula (57).

The proof of this formula for arbitrary values of r and s is again based on the comparison of the analytic structures and periodicity properties of both sides. The constant C_k cannot be fixed by this consideration and in order to find it one should do some additional analysis.

Now with the help of equation (48) we can do the integration with respect to the toron field and obtain

$$\begin{aligned} \int_0^1 \mathrm{d}\tilde{t}_1 \int_0^1 \mathrm{d}\tilde{t}_2 \left| \det \left\| \left(\chi, S_t^{(k)}\right) \right\| \right|^2 &= |C_k|^2 \eta^{6s} \left(\frac{2k}{|\tau|}\right)^{k/2} \frac{1}{\sqrt{2|\tau|}\, L_1^{2k+2s}} \\ &\times \exp\left\{ -\frac{2\pi}{|\tau|} \left[\sum_{i<i'}^{r} (\zeta_i - \zeta_{i'})^2 + \sum_{j<j'}^{s} (\xi_j - \xi_{j'})^2 - \sum_{i=1}^{r}\sum_{j=1}^{s} (\zeta_i - \xi_j)^2 \right] \right\} \\ &\times \frac{\prod_{i<i'}^{r} |\theta_1(z_i - z_{i'})|^2 \prod_{j<j'}^{s} |\theta_1(w_j - w_{j'})|^2}{\prod_{i=1}^{r}\prod_{j=1}^{s} |\theta_1(z_i - w_j)|^2}. \end{aligned} \tag{58}$$

As we see for our aim it is sufficient to know only $|C_k|^2$. In order to find it we may use the properties which the objects entering equation (58) demonstrate under the modular transformation considered in section 3. The lhs of equation (58) is invariant under this transformation, so its rhs should be invariant as well. With the help of equation (26) we find that it will really be the case if

$$|C_k|^2 = \eta^{-(k-1)(k-2)}. \tag{59}$$

Using this expression together with equations (58), (33) and (31) in equation (53) we will get the desired result (50).

The case when $s = 0$ can be considered similarly. Now instead of the matrix (52) we will have a matrix of the zero modes $\left\| \chi_1^{(j)}(x_i) \right\|$ only. To obtain the result in this case we may use the formula

$$\det \left\| T_k^{(j)}(z_i + \bar{t}/k) \right\| = C_k \prod_{i<j}^{k} \theta_1(z_i - z_j)\theta_a \left(\sum_{i=1}^{k} z_i + \bar{t} \right) \tag{60}$$

where $a = 1$ (3) if k is even (odd). This formula can be proved by the same method as formula (57).

5. Conclusions

Many interesting features of the SM on a torus are related to the fact that on the torus one can separate in a simple manner the zero modes from the other degrees of freedom. They need a special treatment in quantum theory and contribute to correlation functions of the fermion fields. The role of the zero modes in the chiral symmetry breaking by an anomaly and in the occurrence of clustering becomes particularly transparent.

The dynamics of the toron field is determined by the action $\Gamma^{(0)}[\mathrm{t}]$, which is induced by the effect of this field on the fermions. It controls infrared singularities. The averaging with respect to the toron field assures a translation-invariant distribution of the symmetry breaking zero modes in the topologically nontrivial sectors.

We see from equation (41) that the torus (finite temperature and finite size) result can be obtained from infinite plane (zero temperature and infinite size) result [7] just by replacing the infinite plane chiral condensate and free massive boson Green's function by their torus counterparts.

Knowledge of the exact expressions of the N-point correlators is necessary to find the finite temperature spectral functions and offers important information related to the symmetry problems [25].

There are several interesting issues possible for further investigation.

One can extend our consideration to the case of the still exactly soluble geometric ($N_f = 2$) [3] and multiflavour (N_f is arbitrary) massless SM [26], where in the spectrum in addition to one massive particle there appears an iso-spin multiplet of massless particles. In this case the factor N_f which appears in the toron action will change the character of the toron integration considerably and hence their dynamical role.

The general formula (41) which we obtained in the present work is extremely useful for the consideration of the two-dimensional QED with one flavour massive fermions (massive SM). This model is not exactly soluble but one can do the perturbation expansion in the fermion mass following the approach developed in [27].

Perturbation theory in fermion masses cannot be employed in the $N_f \geqslant 2$ case as physical quantities are not analytic in fermion mass at $T = 0$ [28, 29]. In this case it can be applied only to the high-temperature regime. In papers [29] QED_2 with massive fermions on a circle has been investigated by the method of Abelian bosonization. We believe that new results in QED_2 with massive N-flavour fermions will help us to understand how the effect of quark masses modifies the vacuum structure, meson masses, mixing and the pattern of chiral symmetry breaking.

Another interesting problem is to consider SM on a torus at finite density [30] and find how the chemical potential will enter into our general formula (41).

A detailed discussion of different limits $L_1, L_2 \to \infty$ is done in the appendix. For an useful discussion related to this problem see [6].

Although the present calculations have been done for a simple two-dimensional Abelian model we hope that they further our intuition needed to understand non-perturbative physics of realistic theory such as QCD [31]. They could be useful for comparison with the results obtained by the authors who do lattice simulations of the SM [9].

Recently, a systematic comparison between SM on a torus in the present Euclidean (path integral) approach and SM on a circle in a Hamiltonian (canonical) approach [32, 33] has been fulfilled [11]. It is worthwhile to mention that the general formula (41) can also be obtained in the second approach.

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Appendix. Thermodynamic and zero-temperature limits

In the *thermodynamic limit* when L_1 tends to infinity or $|\tau| \to 0$ ($L_2 \equiv \beta = T^{-1}$) we will get from $\overline{G}_m(x)$ the propagator $D_{m,\beta}(x)$ of the free massive particles at finite temperatures in infinite space, which can be expressed in three different forms:
as power series:

$$\overline{G}_m(x) \approx D_{m,\beta}(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} K_0\left(m\sqrt{(x_2 - n\beta)^2 + x_1^2}\right) \tag{A.1}$$

where $K_0(x)$ is the McDonald function,

$$D_{m,\beta}(x) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\tilde{E}(n)|x_1| - 2\pi \mathrm{i} n \frac{x_2}{\beta}}}{\tilde{E}(n)} \tag{A.2}$$

where $\tilde{E}(n) = \sqrt{4\pi^2 n^2 \beta^{-2} + m^2}$,
and as an integral:

$$D_{m,\beta}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kx_1} \frac{\cosh\left[\frac{\beta}{2}\sqrt{m^2 + k^2}\left(1 - 2\frac{|x_2|}{\beta}\right)\right]}{\sqrt{k^2 + m^2}\, \sinh \frac{\beta}{2}\sqrt{k^2 + m^2}}. \tag{A.3}$$

In order to find the propagator of free massless particles in the infinite space but at finite temperature we can consider expression (26) in the thermodynamic limit. Taking into account the following approximate expressions for the θ_1- and η-functions in this limit

$$\theta_1(z) \approx \frac{2}{\sqrt{|\tau|}}\, \mathrm{e}^{-\frac{\pi}{4|\tau|}} \sinh\frac{\pi z}{|\tau|} \qquad \eta(\tau) \approx \frac{\mathrm{e}^{-\frac{\pi}{12|\tau|}}}{\sqrt{|\tau|}} \tag{A.4}$$

we obtain

$$G_0(x) \approx D_{0,\beta}(x) = -\frac{1}{2\pi} \ln\left(\mu\beta\sqrt{\cosh\frac{2\pi x_1}{\beta} - \cos\frac{2\pi x_2}{\beta}}\right) \tag{A.5}$$

where μ is an infinitesimal mass needed for infrared regularization. Note that the same expression can easily be obtained if we consider the limit $m \to 0$ in expression (A.2).

The limiting expression for the chiral condensate equation (33)

$$\langle\bar{\psi}(x)P_\pm\psi(x)\rangle = -\frac{m}{4\pi}\mathrm{e}^{\gamma}\,\mathrm{e}^{-2\sum_{l=1}^{\infty}K_0(lm\beta)} = -\frac{m}{4\pi}\mathrm{e}^{\gamma}\,\mathrm{e}^{-\int_{-\infty}^{\infty}\mathrm{d}k\frac{1}{(\mathrm{e}^{\beta\sqrt{m^2+k^2}}-1)\sqrt{m^2+k^2}}} \tag{A.6}$$

where $\gamma = 0.577\,21\ldots$ is the Euler constant, was obtained in the papers [4, 6].

In this limit for the first leading term of the propagator of fermions in the external toron field (see equation (20)) we get

$$\left(S_t^{(0)}(x)\right)_{12} \approx \frac{\mathrm{e}^{-\frac{2\pi z}{|\tau|}\left(\tilde{t}_2-\frac{1}{2}\right)+\frac{2\pi\mathrm{i}\zeta}{|\tau|}\tilde{t}_2}}{2L_2\,\sinh\frac{\pi z}{|\tau|}} \tag{A.7}$$

where we used equation (A.4) and the approximation valid for $L_1 \to \infty$

$$\theta_1(z+\bar{t}) \approx \frac{1}{\sqrt{|\tau|}}\,\mathrm{e}^{-\frac{\pi}{|\tau|}(\bar{t}-\frac{1}{2})^2-\frac{2\pi z}{|\tau|}(\bar{t}-\frac{1}{2})}. \tag{A.8}$$

(it is assumed that $0 < \tilde{t}_2 < 1$.)

Note that the proof of the formula (41) for $r = s$ in this limiting case, where the formulae (A.7) and (A.8) should be used, could be based on the formula

$$\det\left\|\frac{1}{f(v_i-u_j)}\right\| = (-1)^{\frac{r(r-1)}{2}}\frac{\prod_{i<j}^{r} f(v_i-v_j)f(u_i-u_j)}{\prod_{i,j}^{r} f(v_i-u_j)} \tag{A.9}$$

where $f(z) = \sinh\pi z/|\tau|$ which follows directly from the Cauchy determinant formula (47) if one takes $z = \mathrm{e}^{2\mathrm{i}\pi v}$ and $w = \mathrm{e}^{2\mathrm{i}\pi u}$.

Of course, equation (A.9) is a particular case of formula (46), which occurs if $L_1 \to \infty$.

In order to consider the thermodynamic limit in the sectors with nontrivial topological charge k we should know the limiting expressions for the fermionic Green's function in these sectors, equation (21), and the fermionic zero modes (11).

We can easily obtain the first leading terms of both objects in this limit. Using the fact that for $|\tau| \to 0$

$$\theta_3(z) = \frac{1}{\sqrt{|\tau|}}\,\mathrm{e}^{-\frac{\pi z^2}{|\tau|}}\left(1 + 2\mathrm{e}^{-\frac{\pi}{|\tau|}}\cosh\frac{2\pi z}{|\tau|} + \cdots\right) \tag{A.10}$$

we get for $S_t^{(k)}(x, y)$

$$S_t^{(k)}(x,y) \approx S_t^{(0)}(x,y)|_{L_1\to\infty} \tag{A.11}$$

since, e.g.

$$q^{(k)}(z) = \theta_3\left(z\,\Big|\,\frac{\tau}{k}\right) = \sqrt{\frac{k}{|\tau|}} + \cdots \tag{A.12}$$

and $S_t^{(0)}(x,y)|_{L_1\to\infty}$ is given in equation (A.7).

For the fermionic zero modes (11) (for example, with the choice (14)) we obtain: when $k = 1$

$$\chi_1^{(1)}(x) \approx \left(\frac{2}{|\tau|}\right)^{1/4} \frac{1}{|\tau|^{1/2}L_1}\, \mathrm{e}^{\frac{2\pi \mathrm{i}\zeta}{|\tau|}\tilde{t}_2 - \frac{2\pi z}{|\tau|}\tilde{t}_2 - \frac{\pi}{|\tau|}\tilde{t}_2^2 + \mathrm{i}\pi\tilde{t}_2\tilde{t}_1} \tag{A.13}$$

for $0 < \tilde{t}_2 < \frac{1}{2}$ and

$$\chi_1^{(1)}(x) \approx \left(\frac{2}{|\tau|}\right)^{1/4} \frac{1}{|\tau|^{1/2}L_1}\, \mathrm{e}^{\frac{2\pi \mathrm{i}\zeta}{|\tau|}\tilde{t}_2 - \frac{2\pi z}{|\tau|}(\tilde{t}_2 - 1) - \frac{\pi}{|\tau|}(\tilde{t}_2-1)^2 + \mathrm{i}\pi\tilde{t}_2\tilde{t}_1} \tag{A.14}$$

for $\frac{1}{2} < \tilde{t}_2 < 1$.

When $k \geqslant 2$

$$\chi_k^{(j)}(x) \approx \left(\frac{2k}{|\tau|}\right)^{1/4} \frac{1}{|\tau|^{1/2}L_1}\, \mathrm{e}^{\frac{2\pi \mathrm{i}\zeta}{|\tau|}\tilde{t}_2 - \frac{2\pi z}{|\tau|}[\bar{t}-(j-1)] - \frac{\pi}{k|\tau|}[\bar{t}-(j-1)]^2 - \frac{\mathrm{i}\pi}{k}\tilde{t}_2\tilde{t}_1} \tag{A.15}$$

for $1 \leqslant j \leqslant \frac{k}{2} + 1$ and

$$\chi_k^{(j)}(x) \approx \left(\frac{2k}{|\tau|}\right)^{1/4} \frac{1}{|\tau|^{1/2}L_1}\, \mathrm{e}^{\frac{2\pi \mathrm{i}\zeta}{|\tau|}\tilde{t}_2 - \frac{2\pi z}{|\tau|}[\bar{t}-(j-1)] - \frac{\pi}{k|\tau|}[\bar{t}-(j-1)]^2 - \frac{\mathrm{i}\pi}{k}\tilde{t}_2\tilde{t}_1}\, \mathrm{e}^{-\frac{2\pi}{|\tau|}\left[\bar{t}-(j-1)+\frac{k}{2}\right] - \frac{2\pi z k}{|\tau|}} \tag{A.16}$$

for $\frac{k}{2} + 1 < j \leqslant k$.

But keeping only the first leading terms allows us to explicitly check formulae (57) and (60) only for $k = 1$ and $k = 2$ in this limit. For $k \geqslant 3$ in order to do this we should keep the appropriate number of higher-order terms in the expansions of the fermionic Green's functions $S_t^{(k)}(x,y)$ and the fermionic zero modes, because of the nature of the expansions of θ_1 and θ_3 functions given in equations (A.4) and (A.10). This makes explicit checking rather difficult.

Now let us consider the *zero-temperature limit* $L_2 \to \infty$ ($|\tau| \to \infty$) ($L_1 \equiv L$). The expressions for the propagator of the free massive and massless bosons on a circle in Euclidean time (in the infinite interval) one can get from equations (A.1)–(A.3) and (A.5) by replacing β by L and interchanging $x_1 \leftrightarrow x_2$. Of course, the propagator of the free massless bosons in this limit can again be obtained directly from equation (26) using the approximate expressions for the θ_1 function in the limit $|\tau| \to \infty$

$$\theta_1(z) \approx 2\mathrm{e}^{-\frac{\pi|\tau|}{4}} \sin \pi z \qquad \eta(\tau) \approx \mathrm{e}^{-\frac{\pi|\tau|}{12}}. \tag{A.17}$$

The limiting expression for the chiral condensate was obtained in papers [32], and has the form given in equation (A.6) where again β is replaced by L.

For the first leading term of the propagator of fermions in the external toron field in zero-temperature limit we get

$$\left(S_t^{(0)}(x)\right)_{12} \approx \frac{\mathrm{e}^{2\pi \mathrm{i} z}}{2L \sin \pi z} \tag{A.18}$$

where equation (A.17) and the approximation valid for $L_2 \to \infty$

$$\theta_1(z + \bar{t}) \approx -\mathrm{i}\,\mathrm{e}^{-\frac{\pi|\tau|}{4}}\, \mathrm{e}^{\mathrm{i}\pi(z+\bar{t})} \tag{A.19}$$

are used. We see that in this case the first leading term does not depend on toron field t and the integration with respect to it in equation (49) becomes trivial (note that in this limit $|\theta_1(t)|^2 \approx \mathrm{e}^{-\frac{\pi|\tau|}{2}+2|\tau|\tilde{t}_1}$).

As in the previous case the proof of the formula (41) for $k = 0$ in zero-temperature limit can be based on the formula (A.9) which is also valid if $f(z) = \sin \pi z$.

For nontrivial topological sectors comments similar to the case $L_1 \to \infty$ could be made.

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